

Transverse instability of gravity–capillary solitary waves

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Abstract Gravity–capillary solitary waves of depression that bifurcate at the minimum phase speed on water of finite or infinite depth, while stable to perturbations along the propagation direction, are found to be unstable to transverse perturbations on the basis of a long-wave stability analysis. This suggests a possible generation mechanism of the new class of gravity–capillary lumps recently shown to also bifurcate at the minimum phase speed.

Keywords Gravity–capillary solitary waves · Long-wave stability analysis · Perturbation method · Total energy of a solitary wave · Transverse instability

1 Introduction

The stability of plane solitary waves to perturbations transverse to the direction of propagation was examined first by Kadomtsev and Petviashvili [1] on the basis of a generalization of the classical Korteweg–de Vries (KdV) equation allowing for weak transverse variations. This model equation, now known as the Kadomtsev–Petviashvili (KP) equation, predicts that KdV gravity–capillary solitary waves on shallow water are unstable to transverse perturbations if surface tension is strong enough:

$$T > \frac{1}{3}\rho gh^2, \quad (1.1)$$

where T denotes the coefficient of surface tension, h the water depth, ρ the fluid density and g the gravitational acceleration. When this condition is satisfied, moreover, the KP equation admits fully localized solitary-wave solutions, commonly referred to as ‘lumps’, which thus become the asymptotic states of the initial-value problem in two spatial dimensions [2].

In more recent work, Bridges [3] obtained a condition for transverse instability to long-wave perturbations of solitary-wave solutions of Hamiltonian partial differential equations formulated as multi-symplectic

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systems. When applied to the water-wave problem with gravity and possibly surface tension present, this condition implies instability if

$$\frac{\partial I}{\partial V} < 0, \quad (1.2)$$

where I is a certain quantity related to the total horizontal linear momentum, or impulse, of the solitary wave and V denotes the wave speed. (Bridges [3] refers to I as the impulse but this ignores the circulation of the solitary wave which is non-zero in general; see Sect. 2 below.)

In the case of pure gravity solitary waves, it was shown by Longuet-Higgins [4] that $V\partial I/\partial V = \partial \mathcal{E}/\partial V$, \mathcal{E} being the total energy of the solitary wave, so an alternative form of the instability condition (1.2) is

$$\frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} < 0. \quad (1.3)$$

Moreover, \mathcal{E} is known to be an increasing function of V in the weakly non-linear limit, where the KdV equation applies, and also for finite-amplitude waves with steepness (amplitude-to-depth ratio) below $\epsilon = 0.781$. Hence, according to (1.3), transverse instability of gravity solitary waves first arises when the wave steepness exceeds this critical value which happens to also mark the onset of superharmonic instability to longitudinal perturbations [5].

It is worth noting that (1.2) and (1.3) apply in the long-wave-disturbance limit and provide sufficient, but not necessary, conditions for instability. This is consistent with recent work by Kataoka and Tsutahara [6], who re-visited, using perturbation expansions, the eigenvalue problem governing the transverse instability of gravity solitary waves of the KdV type to long-wave disturbances. While their leading-order solution recovers condition (1.3), at the next order they find that instability in fact sets in at a somewhat lower wave steepness, $\epsilon = 0.713$, than the critical value 0.781 furnished by (1.3); according to the refined stability criterion, therefore, transverse instability arises prior to longitudinal instability.

The present paper is concerned with the transverse instability of gravity–capillary solitary waves. Rather than the classical KdV solitary waves on shallow water, attention is focused on solitary waves of the wavepacket type that are possible on water of finite or infinite depth in the presence of both gravity and surface tension, and have been studied extensively in recent years (see [7] for a review). Solitary waves of this kind bifurcate from linear sinusoidal wavetrains at the minimum gravity–capillary phase speed and, in the small-amplitude limit, resemble wavepackets whose wave envelope and crests travel at the same speed [8,9]. Out of the two symmetric solitary-wave solution branches that bifurcate at the minimum phase speed, the depression branch is stable to longitudinal perturbations [10], so it is natural to inquire into its stability with respect to transverse perturbations.

It is straightforward to show (see Sect. 2) that the form (1.3), in terms of the total energy \mathcal{E} , of the transverse-instability condition (1.2) obtained by Bridges [3] remains valid in the case of gravity–capillary solitary waves as well. This suggests that the solitary waves of interest here, which exist below the minimum gravity–capillary phase speed, are unstable to transverse perturbations, as \mathcal{E} is expected to increase when the wave speed V is decreased. In fact, for a portion of the depression solitary-wave solution branch in deep water for which computations of \mathcal{E} are available [11], it is clear that $\partial \mathcal{E}/\partial V < 0$, implying instability according to (1.3).

Here we examine the stability of gravity–capillary solitary waves of the wavepacket type to long-wave transverse perturbations and compute the instability growth rate in a systematic way, using the expansion procedure of Kataoka and Tsutahara [6] with the added effect of surface tension. The entire branch of depression solitary waves on water of finite or infinite depth, while stable to longitudinal perturbations, turns out to be transversely unstable, and the instability growth rate increases with \mathcal{E} .

In analogy with transversely unstable KdV solitary waves that give rise to KP lumps in the high-surface-tension regime (1.1), it is likely that the instability discussed here also results in the formation of gravity–capillary lumps, but of the type recently found by Kim and Akylas [12] and Parau et al. [13].

2 Preliminaries

It is customary in studies of gravity–capillary solitary waves of the wavepacket type (e.g., [8, 10, 14, 15]) to introduce non-dimensional variables such that the wave speed c is normalized to 1 and trace solitary-wave solution branches in terms of

$$\alpha = \frac{gT}{\rho c^4} \tag{2.1}$$

and an additional parameter, such as

$$H = \frac{h\rho c^2}{T}, \tag{2.2}$$

that involves the water depth. For the purpose of discussing the transverse instability of these solitary waves, however, it proves more convenient to use as characteristic length and time scale, respectively, $(T/\rho g)^{1/2}$ and $(T/\rho g^3)^{1/4}$, which do not depend on c ; solitary-wave solution branches are thus traced via the dimensionless wave speed

$$V = c \left(\frac{\rho}{gT} \right)^{1/4} = \alpha^{-1/4}, \tag{2.3}$$

and the water-depth parameter

$$D = h \left(\frac{\rho g}{T} \right)^{1/2} = H\alpha^{1/2}. \tag{2.4}$$

Gravity–capillary solitary waves of the wavepacket type arise when the minimum phase speed occurs at a finite wavenumber [8, 9], and this is possible when condition (1.1) is not met, implying $D > \sqrt{3}$.

As remarked earlier, the total energy $\mathcal{E}(V, D)$ of a solitary wave plays an important part in the stability analysis. In terms of the velocity potential $\bar{\phi}(\theta, z; V, D)$ and the free-surface elevation $z = \bar{\eta}(\theta; V, D)$, where $\theta = x - Vt$, associated with a solitary wave travelling along x with speed V on water of depth D ($-D < z < \bar{\eta}$), \mathcal{E} is given by

$$\mathcal{E} = \mathcal{T} + \mathcal{V}_G + \mathcal{V}_T, \tag{2.5}$$

where

$$\mathcal{T} = \frac{1}{2} \int_{-\infty}^{\infty} d\theta \int_{-D}^{\bar{\eta}} (\bar{\phi}_\theta^2 + \bar{\phi}_z^2) dz \tag{2.6}$$

denotes the kinetic energy,

$$\mathcal{V}_G = \frac{1}{2} \int_{-\infty}^{\infty} \bar{\eta}^2 d\theta \tag{2.7}$$

the gravitational potential energy and

$$\mathcal{V}_T = \int_{-\infty}^{\infty} \left\{ \left(1 + \bar{\eta}_\theta^2 \right)^{1/2} - 1 \right\} d\theta \tag{2.8}$$

the potential energy due to surface tension.

In addition, the total horizontal momentum, or impulse, $\mathcal{I}(V, D)$ of the solitary wave is given by

$$\mathcal{I} = \int_{-\infty}^{\infty} d\theta \int_{-D}^{\bar{\eta}} \bar{\phi}_\theta dz. \tag{2.9}$$

From (2.9), upon integrating by parts in θ , one then has

$$\mathcal{I} = I + DC, \tag{2.10}$$

where

$$C = \bar{\phi} \Big|_{\theta=-\infty}^{\theta=\infty} \quad (2.11)$$

is the circulation of the solitary wave [4] and

$$I = - \int_{-\infty}^{\infty} d\theta \bar{\eta}_\theta \bar{\phi} \Big|_{z=\bar{\eta}} \quad (2.12)$$

is the quantity that enters the instability condition (1.2) derived by Bridges [3]. Note that, in water of finite depth, $C \neq 0$ so I is distinct from the true impulse in general.

Differentiating expression (2.12) for I with respect to V and combining the result with (2.10), we have

$$\frac{\partial I}{\partial V} = \frac{\partial \mathcal{I}}{\partial V} - D \frac{\partial C}{\partial V} = \int_{-\infty}^{\infty} d\theta (\bar{\eta}_V \bar{\phi}_\theta - \bar{\eta}_\theta \bar{\phi}_V) \Big|_{z=\bar{\eta}}. \quad (2.13)$$

For pure gravity solitary waves, it is known from Longuet-Higgins [4] that

$$\frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} = \frac{\partial \mathcal{I}}{\partial V} - D \frac{\partial C}{\partial V}, \quad (2.14)$$

so condition (1.2) can be replaced by (1.3), as indicated earlier. This remains true, however, when surface tension is also present, since it can be readily verified by differentiating (2.5) with respect to V that

$$\frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} = \int_{-\infty}^{\infty} d\theta (\bar{\eta}_V \bar{\phi}_\theta - \bar{\eta}_\theta \bar{\phi}_V) \Big|_{z=\bar{\eta}}. \quad (2.15)$$

Hence, according to (2.13),

$$\frac{\partial I}{\partial V} = \frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} \quad (2.16)$$

so condition (1.2) is entirely equivalent to (1.3) in the case of gravity–capillary solitary waves as well.

3 Long-wave stability analysis

To diagnose the stability to long transverse perturbations of gravity–capillary solitary waves of the wave-packet type, one may appeal directly to condition (1.2), or equivalently condition (1.3) in view of (2.16). In addition, it is possible to deduce the associated instability growth rate in the long-wave limit from the theory of Bridges [3]. For the latter purpose, rather than specializing the general formalism to the problem at hand, we find it more instructive to directly tackle the stability eigenvalue problem via a long-wave expansion procedure analogous to the one followed by Kataoka and Tsutahara [6] for pure gravity solitary waves. A similar approach was taken by Kim and Akylas [16] for the transverse instability of solitary waves of the Benjamin equation.

Briefly, assuming that infinitesimal perturbations with wavenumber μ in the transverse (y -) direction and growth rate λ are present in the free-surface elevation η and the potential ϕ ,

$$\eta = \bar{\eta} + \hat{\eta}(\theta) e^{i\mu y + \lambda t}, \quad \phi = \bar{\phi} + \hat{\phi}(\theta, z) e^{i\mu y + \lambda t}, \quad (3.1)$$

we linearize the governing equations about the underlying solitary-wave state. The perturbation eigenfunctions $\hat{\phi}$ and $\hat{\eta}$ then satisfy the Helmholtz equation

$$\hat{\phi}_{\theta\theta} + \hat{\phi}_{zz} = \mu^2 \hat{\phi} \quad (-D < z < \bar{\eta}), \quad (3.2)$$

subject to the bottom condition

$$\hat{\phi}_z = 0 \quad (z = -D) \quad (3.3)$$

and the following two conditions on the free surface of the solitary wave:

$$\mathcal{L}_1(\hat{\phi}, \hat{\eta}) = -\lambda \hat{\eta} \quad (z = \bar{\eta}), \quad (3.4)$$

$$\mathcal{L}_2(\hat{\phi}, \hat{\eta}) = -\lambda\hat{\phi} - \mu^2 \frac{\hat{\eta}}{(1 + \bar{\eta}_\theta^2)^{1/2}} \quad (z = \bar{\eta}), \tag{3.5}$$

where

$$\mathcal{L}_1(\hat{\phi}, \hat{\eta}) \equiv \left(-\frac{\partial}{\partial z} + \bar{\eta}_\theta \frac{\partial}{\partial \theta} \right) \hat{\phi} + \left\{ \frac{d\bar{\phi}_\theta}{d\theta} + (-V + \bar{\phi}_\theta) \frac{\partial}{\partial \theta} \right\} \hat{\eta}, \tag{3.6}$$

$$\mathcal{L}_2(\hat{\phi}, \hat{\eta}) \equiv (-V + \bar{\phi}_\theta) \frac{d\hat{\phi}}{d\theta} + \{(-V + \bar{\phi}_\theta)\bar{\phi}_{\theta z} + \bar{\phi}_z \bar{\phi}_{zz} + 1\} \hat{\eta} - \left\{ \frac{\hat{\eta}_\theta}{(1 + \bar{\eta}_\theta^2)^{3/2}} \right\}_\theta. \tag{3.7}$$

The boundary conditions (3.4) and (3.5) result from linearizing the kinematic and dynamic free-surface conditions, respectively. Compared with [6], the additional effects of surface tension in the dynamic condition are reflected in the last term in (3.7) and the $O(\mu^2)$ term in (3.5).

In the long-wave limit ($\mu \ll 1$), the eigenvalue problem (3.2)–(3.5) is solved by expanding $\hat{\phi}$, $\hat{\eta}$ and the growth rate λ , which acts as the eigenvalue, in powers of μ :

$$\hat{\phi} = \hat{\phi}^{(0)} + \mu\hat{\phi}^{(1)} + \mu^2\hat{\phi}^{(2)} + \dots, \quad \hat{\eta} = \hat{\eta}^{(0)} + \mu\hat{\eta}^{(1)} + \mu^2\hat{\eta}^{(2)} + \dots, \tag{3.8}$$

$$\lambda = \mu\lambda_1 + \mu^2\lambda_2 + \dots. \tag{3.9}$$

As in [6], the leading-order solution is readily shown to be

$$\hat{\phi}^{(0)} = \bar{\phi}_\theta, \quad \hat{\eta}^{(0)} = \bar{\eta}_\theta. \tag{3.10}$$

Proceeding to $O(\mu)$, we observe that $\hat{\phi}^{(1)}$ and $\hat{\eta}^{(1)}$ satisfy the forced problem

$$\hat{\phi}_{\theta\theta}^{(1)} + \hat{\phi}_{zz}^{(1)} = R \quad (-D < z < \bar{\eta}), \tag{3.11}$$

$$\hat{\phi}_z^{(1)} = 0 \quad (z = -D), \tag{3.12}$$

$$\mathcal{L}_1(\hat{\phi}^{(1)}, \hat{\eta}^{(1)}) = r_1 \quad (z = \bar{\eta}), \tag{3.13}$$

$$\mathcal{L}_2(\hat{\phi}^{(1)}, \hat{\eta}^{(1)}) = r_2 \quad (z = \bar{\eta}), \tag{3.14}$$

where

$$R = 0, \quad r_1 = -\lambda_1 \bar{\eta}_\theta, \quad r_2 = -\lambda_1 \bar{\phi}_\theta. \tag{3.15}$$

Since (3.10) is a well-behaved solution of the corresponding homogeneous problem, the forcing terms (3.15) must satisfy a certain condition for the inhomogeneous boundary-value problem (3.11)–(3.14) to have a solution that behaves acceptably as $\theta \rightarrow \pm\infty$. This solvability condition can be derived by forming the inner product of the forced equation (3.11) with the well-behaved solution of the adjoint boundary-value problem, and then making use of Green’s identity along with the boundary conditions (3.12)–(3.14) and the properties of the adjoint solution. As it parallels closely an analogous derivation in [6], here we shall omit the details (see [17]) and simply state the solvability condition:

$$\int_{-\infty}^{\infty} d\theta \int_{-D}^{\bar{\eta}} \bar{\phi}_\theta R \, dz + \int_{-\infty}^{\infty} d\theta (r_1 \bar{\phi}_\theta - r_2 \bar{\eta}_\theta) \Big|_{z=\bar{\eta}} = 0. \tag{3.16}$$

For the forcing terms (3.15), this condition is trivially met and the well-behaved solution of the problem (3.11)–(3.14) turns out to be

$$\hat{\phi}^{(1)} = -\lambda_1 \bar{\phi}_V, \quad \hat{\eta}^{(1)} = -\lambda_1 \bar{\eta}_V. \tag{3.17}$$

At the next order, $\hat{\phi}^{(2)}$ and $\hat{\eta}^{(2)}$ are governed again by a forced problem of the form (3.11)–(3.14) but with forcing terms

$$R = \bar{\phi}_\theta, r_1 = -\lambda_2 \bar{\eta}_\theta + \lambda_1^2 \bar{\eta}_V, r_2 = -\lambda_2 \bar{\phi}_\theta + \lambda_1^2 \bar{\phi}_V - \frac{\bar{\eta}_\theta}{(1 + \bar{\eta}_\theta^2)^{1/2}}. \tag{3.18}$$

Inserting the forcing terms (3.18) in the solvability condition (3.16) now yields a non-trivial result,

$$\lambda_1^2 \int_{-\infty}^{\infty} d\theta (\bar{\eta}_V \bar{\phi}_\theta - \bar{\eta}_\theta \bar{\phi}_V) \Big|_{z=\bar{\eta}} + \int_{-\infty}^{\infty} d\theta \left(\int_{-D}^{\bar{\eta}} \bar{\phi}_\theta^2 dz + \frac{\bar{\eta}_\theta^2}{(1 + \bar{\eta}_\theta^2)^{1/2}} \right) = 0, \quad (3.19)$$

which, taking into account (2.15), furnishes the following expression for the growth rate of long transverse perturbations to leading order in μ :

$$\lambda_1^2 = - \left(\int_{-\infty}^{\infty} d\theta \int_{-D}^{\bar{\eta}} \bar{\phi}_\theta^2 dz + \int_{-\infty}^{\infty} \frac{\bar{\eta}_\theta^2}{(1 + \bar{\eta}_\theta^2)^{1/2}} d\theta \right) / \frac{1}{V} \frac{\partial \mathcal{E}}{\partial V}. \quad (3.20)$$

This confirms that (1.3) is a sufficient condition for transverse instability of solitary waves in the presence of gravity and surface tension.

As remarked earlier, in the case of pure gravity solitary waves, the instability condition (1.3) is first met when the wave steepness exceeds a certain critical value [5]; in order to gain information regarding the transverse instability of less steep solitary waves, it is necessary to carry the long-wave expansions (3.8) and (3.9) to higher order, as was done in [6]. Fortunately, in the problem of interest here, as verified below, the leading-order result (3.20) predicts instability ($\lambda_1^2 > 0$) for the entire solution branch of depression gravity–capillary solitary waves, so no higher-order analysis is needed.

4 Results

We now proceed to verify that $\partial \mathcal{E} / \partial V < 0$ for gravity–capillary solitary waves of depression and compute the associated instability growth rate. To this end, we shall make use of small-amplitude expansions in the vicinity of the bifurcation point, slightly below the minimum phase speed, and fully numerical computations in the finite-amplitude regime.

In terms of the non-dimensional variables used here, for a given value of the water-depth parameter $D > \sqrt{3}$, the minimum gravity–capillary phase speed $V = V_0$ and the corresponding wavenumber k_0 are determined from the equation system

$$V_0^2 = \frac{1 + k_0^2}{k_0} \tanh k_0 D, \quad (4.1)$$

$$\frac{1 - k_0^2}{k_0} \tanh k_0 D - \frac{D(1 + k_0^2)}{\cosh^2 k_0 D} = 0. \quad (4.2)$$

In the deep-water limit ($D \rightarrow \infty$), in particular,

$$V_0 \rightarrow \sqrt{2}, \quad k_0 \rightarrow 1. \quad (4.3)$$

For V slightly less than V_0 , the bifurcating solitary-wave solution branches can be computed using perturbation expansions in terms of $\delta \equiv V_0 - V$ ($0 < \delta \ll 1$), and details are given in [8, 9]. Specifically, in the case of deep water, the leading-order results for the depression solitary-wave branch are

$$\bar{\eta} = - \frac{8}{2^{1/4} \sqrt{11}} \delta^{1/2} \operatorname{sech}(2^{3/4} \delta^{1/2} \theta) \cos \theta + \dots, \quad (4.4)$$

$$\bar{\phi} = - \frac{2^{1/4} 8}{\sqrt{11}} \delta^{1/2} e^z \operatorname{sech}(2^{3/4} \delta^{1/2} \theta) \sin \theta + \dots. \quad (4.5)$$

Inserting (4.4) and (4.5) in (2.6)–(2.8), we find that

$$\mathcal{T} \sim \frac{32}{2^{1/4} 11} \delta^{1/2}, \quad (4.6)$$

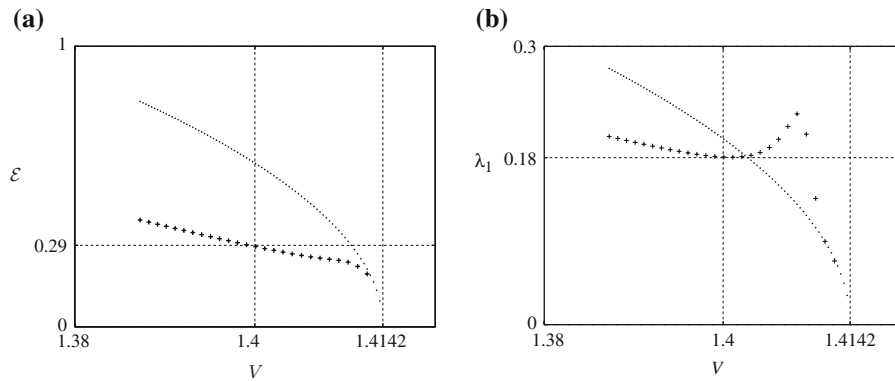


Fig. 1 Comparison between computed results (+) and the asymptotic estimates (4.9)–(4.10) (···) for gravity–capillary solitary waves of speed V near the bifurcation point $V_0 = \sqrt{2}$ in deep water. **(a)** energy; **(b)** instability growth rate λ_1

$$\mathcal{V}_G \sim \frac{16}{2^{1/4}11} \delta^{1/2}, \tag{4.7}$$

$$\mathcal{V}_T \sim \frac{16}{2^{1/4}11} \delta^{1/2}; \tag{4.8}$$

hence,

$$\mathcal{E} \sim \frac{64}{2^{1/4}11} \delta^{1/2}. \tag{4.9}$$

As expected, \mathcal{E} increases as V is decreased below V_0 ($\delta > 0$), so $\partial\mathcal{E}/\partial V < 0$, and small-amplitude solitary waves on deep water are unstable to transverse perturbations. Similarly, from (3.20), making use of (4.4) and (4.5), it follows that

$$\lambda_1 \sim 2^{3/4} \delta^{1/2}. \tag{4.10}$$

Therefore, combining (4.10) and (3.9), the instability growth rate of long-wave transverse perturbations to small-amplitude depression solitary waves, travelling with speed V slightly below the minimum phase speed $V_0 = \sqrt{2}$ in deep water, is given by

$$\lambda \sim 2^{3/4} (V - V_0)^{1/2} \mu, \tag{4.11}$$

$\mu \ll 1$ being the perturbation wavenumber.

Figure 1 compares the asymptotic estimates (4.9) and (4.10) against values of \mathcal{E} and λ_1 , obtained from (2.5)–(2.8) and (3.20) using numerically computed solitary-wave profiles close to the bifurcation point $V_0 = \sqrt{2}$ in deep water. Not unexpectedly, the agreement is good only in the immediate vicinity of the bifurcation point. Numerical computations, nevertheless, indicate that $\partial\mathcal{E}/\partial V$ continues to be negative in the finite-amplitude regime, confirming the presence of instability, as illustrated in Fig. 2 for $D = 2.5$ and $D \rightarrow \infty$. Moreover, the instability growth rate increases rapidly with the solitary-wave steepness, and the instability is stronger (λ_1 is larger) in the deep-water limit ($D \rightarrow \infty$), at least for long-wave perturbations ($\mu \ll 1$).

Solitary-wave profiles and the corresponding values of \mathcal{E} and λ_1 were computed via a Cauchy principal value integral, similar to the one described in [14, 15], combined with Chebyshev spectral collocation (see [17] for details). Table 1 illustrates the convergence of \mathcal{E} as the number of grid points N is varied, for different values of the wave speed V in finite depth ($D = 2.5$) and in deep water ($D \rightarrow \infty$). Our computations of \mathcal{E} also are consistent with those presented in [11, 14] for a portion of the depression solitary-wave branch in deep water. The results shown in Fig. 1 were computed using $N = 2048$, while those shown in Fig. 2 using $N = 512$.

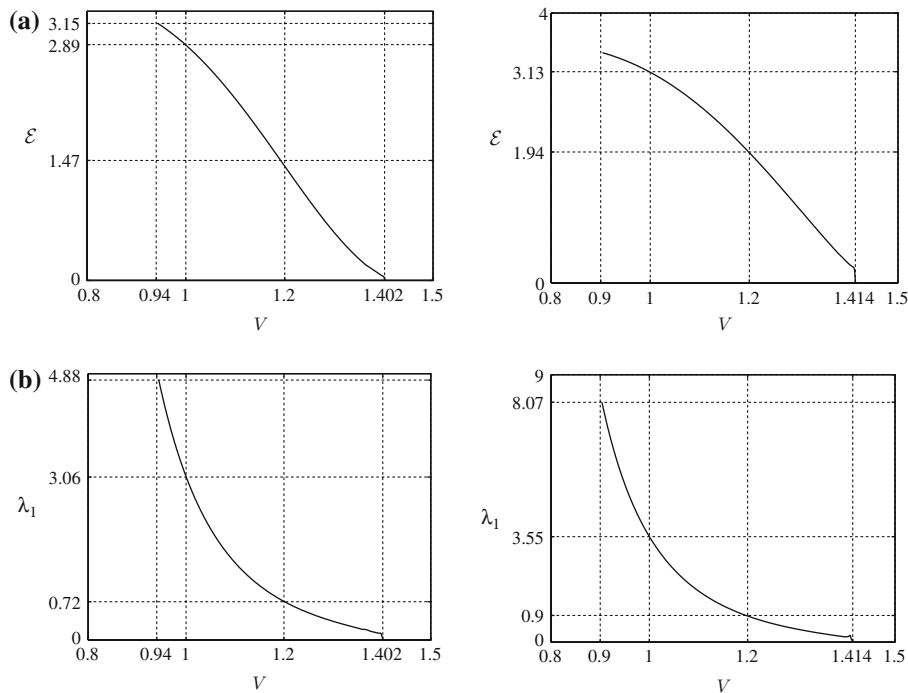


Fig. 2 Energy \mathcal{E} and instability growth rate λ_1 of depression gravity–capillary solitary waves as functions of wave speed V . Left column: finite depth ($D = 2.5$, $V_0 = 1.402$). Right column: infinite depth ($D \rightarrow \infty$, $V_0 = \sqrt{2}$). (a) energy; (b) instability growth rate

Table 1 Convergence of energy \mathcal{E} as the number of grid points N is varied for different values of the wave speed V in finite depth ($D = 2.5$) and in deep water ($D \rightarrow \infty$)

$D = 2.5$				$D \rightarrow \infty$		
$V = 1.0$	$V = 1.2$	$V = 1.4$	N	$V = 1.0$	$V = 1.2$	$V = 1.4$
2.95107	1.46376	0.0230997	512	3.12468	1.93688	0.284594
2.95231	1.46579	0.0232001	1024	3.12650	1.94051	0.285844
2.95269	1.46615	0.0232135	1536	3.12711	1.94171	0.286144
2.95288	1.46642	0.0232221	2048	3.12741	1.94230	0.286265

5 Concluding remarks

The preceding analysis has demonstrated that depression gravity–capillary solitary waves on water of finite or infinite depth, while stable to longitudinal perturbations [10], are unstable to transverse perturbations, similarly to elevation solitary waves of the Benjamin equation [16]. In the latter case, it is known that transverse instability results in the formation of fully locally confined solitary waves (‘lumps’) of elevation, that also bifurcate at the minimum phase speed [16]. In analogy, we expect that the instability discussed here leads to depression lumps of the type recently found in [12, 13].

As it is based on a long-wave expansion, the stability analysis presented here cannot furnish the maximum instability growth rate which corresponds to a finite-wavenumber perturbation ($\mu = O(1)$), in general. This task would require solving the eigenvalue problem (3.2)–(3.5) by numerical means. We remark that, for waves in the gravity–capillary regime, viscous effects may not be negligible, and, in order to assess the physical significance of transverse instability, it would be useful to compare the maximum instability growth

rate against the decay rate caused by viscous dissipation, as was done in [10] for longitudinal perturbations. Based on the results shown in Fig. 2 for long-wave transverse perturbations, however, it would appear that transverse instability is quite strong, particularly for steep solitary waves, and warrants further theoretical and experimental study.

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